Published by Institute of Physics Publishing for SISSA/ISAS
Received: *December 19, 2002*
Revised: *February 26, 2003*
Accepted: *February 28, 2003*

An M-theory solution generating technique and $\mathrm{SL}(2,\mathbb{R})$

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Abstract: In this paper we generalize the $O(p+1,p+1)$ solution generating technique (this is a method used to deform Dp-branes by turning on a NS-NS B-field) to M-theory, in order to be able to deform M5-brane supergravity solutions directly in eleven dimensions, by turning on a non zero three form A. We find that deforming the M5-brane, in some cases, corresponds to performing certain $\mathrm{SL}(2,\mathbb{R})$ transformations of the Kähler structure parameter for the three-torus, on which the M5-brane has been compactified. We show that this new M-theory solution generating technique can be reduced to the $O(p+1,p+1)$ solution generating technique with $p=4$. Further, we find that it implies that the open membrane metric and generalized noncommutativity parameter are manifestly deformation independent for electric and light-like deformations. We also generalize the $O(p+1,p+1)$ method to the type IIA/B NS5-brane in order to be able to deform NS5-branes with RR three and two forms, respectively. In the type IIA case we use the newly obtained solution generating technique and deformation independence to derive a covariant expression for an open D2-brane coupling, relevant for OD2-theory.

Keywords: M-Theory, p-branes, String Duality.

http://jhep.sissa.it/archive/papers/jhep022003060/jhep022003060.pdf

JHEP02(2003)060

Contents

1. Introduction

Recently there has been a lot of interest in theories with noncommutativity (see e.g., [1]–[20]). Using the AdS/CFT correspondence these theories can be studied using supergravity duals (see e.g., [4]–[6], [9, 18, 19] and [21]–[23]). These supergravity duals are obtained by taking a near horizon limit of a supergravity solution corresponding to some bound state. To obtain the relevant supergravity solution (bound state), one can either solve the equations of motion [24, 25], or start with a known brane solution and use some kind of solution generating technique [4, 5, 22, 18]. One solution generating technique which has been very useful is the so called $O(p+1,p+1)$ method [22, 18, 23, 26], which uses elements of the T-duality group to generate bound states (not including the NS5-brane) in type IIA/B supergravity. This method can be seen as deforming D-branes by turning on an NS-NS two form B. An important consequence of this solution generating technique is that it leaves the open string metric and coupling constant invariant and shifts the noncommutativity parameter by a constant [18] (see also [27]). This is referred to as deformation independence of open string data. In [28] the concept of deformation independence was used to derive the open membrane metric and generalized noncommutativity parameter in eleven dimensions, see also [29, 30].

In this paper we are going to generalize the $O(p+1,p+1)$ method to M-theory in order to obtain a formula for deforming M5-branes with a three form directly in eleven dimensions. We also generalize the $O(p+1,p+1)$ method in order to deform the NS5-brane in type IIA/B with a RR three form and RR two form, respectively. These solution generating techniques can not be obtained as easily as the $O(p+1,p+1)$ method since they must include U-duality transformations and not only T-duality transformations. Here, we will obtain them in a more indirect way and check that they are consistent, e.g., we show that the eleven-dimensional supergravity 'tensor' equation of motion is satisfied and that the M5-brane method when reduced to ten dimensions gives the $O(p+1,p+1)$ method for $p=4$. However, in the type IIB NS5-brane case we show how the deformed (D1) metric and RR two form can be derived using certain projective transformations of a tensor $F_{\mu\nu}$ built from the metric and RR two form, similar to how the metric and NS-NS two form are derived when deforming Dp-branes with the $O(p+1,p+1)$ method.

The main reason we derive these solution generating techniques is not in order to be able to obtain new bound states, but rather to obtain more information about symmetries of M-theory, U-duality and how the metric and three form transform under deformations of the M5-brane. In this paper we find that deforming M5-branes in some cases correspond to certain $\mathrm{SL}(2,\mathbb{R})$ transformations of the complex scalar E, which is a certain combination of the determinant of the metric and the three form.[1] Unfortunately we do not obtain a *complete* understanding of how the metric and three form transform under a deformation of the M5-brane. This will be discussed further in a future paper [33]. A more complete understanding would be important to obtain since this might, e.g., give further important information about the open membrane metric, which is invariant under these kinds of transformation (in the electric and light-like cases). However, it is possible that the results obtained in this paper can give some hints.

In section 2 we give a short introduction to the $O(p+1,p+1)$ method, which is of relevance for the rest of the paper. This is followed in section 3.1 by a generalization of the $O(p+1,p+1)$ method to M-theory. In the rest of section 3 we perform several consistency checks in eleven dimensions. Next, in section 4 we reduce the M-theory method to ten dimensions and show that it gives the $O(p+1,p+1)$ method for $p=4$ and rank 2 NS-NS B-field. We further show that reducing the M-theory method transverse to the deformation directions, leads to a method for deforming a type IIA D4-brane with a RR three form. In section 5 we obtain a solution generating technique for deforming a type IIA/B NS5-branes with a RR three or two form, respectively. We also show that the two cases are T-dual to each other. Further, for the type IIA case, using the newly obtained solution generating technique and deformation independence we derive a covariant expression for

[1] The complex scalar E is the Kähler structure parameter for a three-torus, which the M5-brane has been compactified on. For related results we refer to [31, 32].

an open D2-brane coupling, relevant for OD2-theory [13]. Next, in section 6 we show that in certain cases deforming the M5/NS5-branes involves certain $\mathrm{SL}(2,\mathbb{R})$ transformations of the Kähler structure parameter E. We end with some conclusions in section 7.

2. The $O(p+1,p+1)$ method

In this section we give a short review of the $O(p+1,p+1)$ solution generating technique. For more details see [22].

For a NS-NS B-field deformation of a general Dp–brane one first T-dualizes in the directions where one wants to turn on NS-NS fluxes, and then one shifts the B-field with a constant in these directions. After this one T-dualizes in the directions where one has turned on the constant B-field. In a more precise language, the deformation with constant parameter $\theta_{\mathrm{s}}^{\mu\nu}$ is generated by the following $O(p+1,p+1)$ T-duality group element[2]

$$\Lambda = \Lambda_0 \ldots \Lambda_p \Lambda_{-\theta_{\mathrm{s}}} \Lambda_p \cdots \Lambda_0 = J \Lambda_{-\theta_{\mathrm{s}}} J = \Lambda_{\theta_{\mathrm{s}}}^T = \begin{pmatrix} 1 & 0 \\ \theta_{\mathrm{s}} & 1 \end{pmatrix}, \tag{2.1}$$

where $\theta_{\mathrm{s}}^{\mu\nu}$ is dimensionless and carries indices upstairs since it starts life on the T-dual world volume [22]. In (2.1) above, Λ_i $(i = 0, \ldots, p)$ corresponds to a T-duality transformation in the i:th direction, while $\Lambda_{-\theta_{\mathrm{s}}}$ corresponds to a constant shift in B_2 (i.e., a gauge transformation).

Starting with a Dp-brane solution[3]

$$\begin{aligned} ds^2 &= g^{(\mathrm{s})}_{\mu\nu} dx^\mu dx^\nu + g^{(\mathrm{s})}_{mn} dx^m dx^n \,, \qquad e^{2\phi} = g^2 F \,, \\ gC &= \omega dx^0 \wedge \cdots \wedge dx^p + \gamma_{7-p} \,, \end{aligned} \tag{2.2}$$

where F is some function, $\omega = g e^{-\phi} \sqrt{-g^{(\mathrm{s})}_{(\mu\nu)}}$, $(g^{(\mathrm{s})}_{(\mu\nu)} = \det g^{(\mathrm{s})}_{\mu\nu})$ due to the zero force condition, g is the closed string coupling constant and x^μ, $\mu = 0, \ldots, p$, are coordinates in the brane directions, while x^m, $m = p+1, \ldots, 9$, are coordinates in the transverse directions. Also, γ_{7-p} is a transverse form, i.e., $i_\mu \gamma_{7-p} = 0$, where i_μ denotes the inner product with the vector field associated with x^μ. We note that it is only possible to deform (i.e., turn on a non-zero NS-NS two form B) the Dp-brane in those directions in which we can use T-duality. This is a constraint on (2.2).

For the NS-NS fields $g^{(\mathrm{s})}_{\mu\nu}$ and $B_{\mu\nu}$, the transformations in (2.1) imply that the tensor $E_{\mu\nu} = g^{(\mathrm{s})}_{\mu\nu} + B_{\mu\nu}$ transforms by the following projective transformation [22, 18] (Note that in (2.2) $B_{\mu\nu} = 0$)

$$\tilde{E}_{\mu\nu} = \left(\frac{E}{\theta_{\mathrm{s}} E + 1} \right)_{\mu\nu} = \left(\frac{g^{(\mathrm{s})}(1 - \theta_{\mathrm{s}} g^{(\mathrm{s})})}{(1 + \theta_{\mathrm{s}} g^{(\mathrm{s})})(1 - \theta_{\mathrm{s}} g^{(\mathrm{s})})} \right)_{\mu\nu} . \tag{2.3}$$

[2] See [22, 18] for conventions and definitions of the various elements of $O(p+1,p+1)$ appearing in the following discussion.

[3] We are using multi-form notation such that C is a sum of forms while B (see below) has fixed rank 2.

Now using (2.3) and how the dilaton and the RR fields transform (see [22, 18]) we obtain the following deformed Dp-brane configuration:

$$
\begin{aligned}
\tilde{g}^{(\mathrm{s})}_{\mu\nu} &= g^{(\mathrm{s})}_{\mu\rho}\left[(1-(\theta_{\mathrm{s}})^2)^{-1}\right]^{\rho}{}_{\nu}\,, \qquad \tilde{g}^{(\mathrm{s})}_{mn} = g^{(\mathrm{s})}_{mn}\,,\\
\tilde{B}_{\mu\nu} &= -g^{(\mathrm{s})}_{\mu\rho}\theta^{\rho\sigma}_{\mathrm{s}}g^{(\mathrm{s})}_{\sigma\lambda}\left[(1-(\theta_{\mathrm{s}})^2)^{-1}\right]^{\lambda}{}_{\nu}\,,\\
e^{2\tilde{\phi}} &= \frac{e^{2\phi}}{\sqrt{\det(1-(\theta_{\mathrm{s}})^2)}} = e^{2\phi}\left(\frac{\det\tilde{g}}{\det g}\right)^{1/2},\\
g\tilde{C} &= e^{-\frac{1}{2}\tilde{B}_{\mu\nu}dx^{\mu}\wedge dx^{\nu}}(\omega e^{\frac{1}{2}\theta^{\mu\nu}_{\mathrm{s}}i_{\mu}i_{\nu}}dx^0\wedge\cdots\wedge dx^p+\gamma_{7-p})\,,
\end{aligned}
\tag{2.4}
$$

where

$$
(\theta^2_{\mathrm{s}})^{\mu}{}_{\nu} = \theta^{\mu\nu'}_{\mathrm{s}}g^{(\mathrm{s})}_{\nu'\rho}\theta^{\rho\sigma}_{\mathrm{s}}g^{(\mathrm{s})}_{\sigma\nu}\,. \tag{2.5}
$$

There are two types of deformations that are possible: θ^{0i} and θ^{ij}, where $i,j = 1,2,\ldots,p$. The first one is called 'electric' since we mix the time direction with a spatial direction, while the second is called 'magnetic' since the time direction is not included. Further, we note that the deformed solution (2.4) satisfies the zero force condition, because the zero force condition is satisfied by the undeformed solution (2.2), see [22, 18]. The deformed solution also preserves the same amount of supersymmetry as the undeformed solution.

Before we end this review section we note that there is a very simple relation between the open string metric, noncommutativity parameter and the closed string metric and NS-NS B-field. Defining

$$
\tau^{\mu\nu} = G^{\mu\nu}_{(\mathrm{s})} + \frac{\Theta^{\mu\nu}}{\alpha'}\,, \tag{2.6}
$$

it is easy to obtain that E and τ are related through

$$
\tau^{\mu\nu} = (E^{-1})^{\mu\nu}\,. \tag{2.7}
$$

In later sections we will see that similar relations hold also for open D-brane data and closed D-brane data, see in particular section 5.

3. An M-theory solution generating technique and some tests

In this section we will argue for the existence of an M-theory solution generating technique that can be used to deform M5-branes with a non-zero three form A, which obeys a non-linear self-duality equation. As we will see below, this is (as expected) the same non-linear self-duality equation as the gauge invariant M5-brane world volume three form $\mathcal{H}$ satisfies.

In section 3.1 we *conjecture* the exact form of this solution generating technique, while in the following subsections we test the conjecture in eleven dimensions.

3.1 An M-theory solution generating technique

Here we will generalize the $O(p+1,p+1)$ method to eleven dimensions, in order to deform M5-branes with a non-zero three form A. We start with the following (general) M5-brane solution:[4]

$$
\begin{aligned}
ds^2 &= g_{\mu\nu}dx^\mu dx^\nu + g_{mn}dx^m dx^n\,, \qquad \mu,\nu = 0,1,\ldots,5\,, \qquad m,n = 6,7,8,9,10\,,\\
A_6 &= \omega dx^0\wedge\cdots\wedge dx^5\,, \qquad A_3 = \gamma_3\,,
\end{aligned}
\tag{3.1}
$$

where γ_3 is a transverse three form dual to the six form (i.e., $*d\gamma_3 = dA_6$), while $\omega = \sqrt{-g_{(\mu\nu)}}$ due to the zero force condition and the metric is assumed to be diagonal.

An important difference between the $O(p+1,p+1)$ method and a method for deforming M5-branes is that the former can be derived from the T-duality group, while the latter does not seem to be possible to derive because we lack a microscopic formulation of M-theory. We will therefore conjecture the exact form of the M-theory solution generating technique and test this conjecture both directly in eleven dimensions and show that it reduces to known results in ten dimensions. We now conjecture that the generalization of (2.4) to deformations of the eleven-dimensional M5-brane with a non-zero three form A, is given by

$$
\begin{aligned}
\tilde g_{\mu\nu} &= \Big[\det\Big(1+\frac{1}{2}(\theta)^2\Big)\Big]^{1/9} g_{\mu\rho}\Big[\Big(1+\frac{1}{2}(\theta)^2\Big)^{-1}\Big]^{\rho}{}_{\nu}\,, \qquad \tilde g_{mn} = \Big[\det\Big(1+\frac{1}{2}(\theta)^2\Big)\Big]^{1/9} g_{mn}\,,\\
\tilde A_3 &= \tilde A_{3\mathrm{a}} + \tilde A_{3\mathrm{b}} + \gamma_3\,,\\
\tilde A_6 &= A_6 + \frac{1}{2}\tilde A_{3\mathrm{a}}\wedge\tilde A_{3\mathrm{b}} + \frac{1}{2}(\tilde A_{3\mathrm{a}} + \tilde A_{3\mathrm{b}})\wedge\gamma_3\,,
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde A_{3\mathrm{a}} &= \frac{1}{6}\tilde A^{3\mathrm{a}}_{\mu\nu\rho}dx^\mu\wedge dx^\nu\wedge dx^\rho\,, \qquad \tilde A^{3\mathrm{a}}_{\mu\nu\rho} = -g_{\mu\rho'}g_{\sigma\nu}\theta^{\rho'\sigma\sigma'}g_{\sigma'\lambda}\Big[\Big(1+\frac{1}{2}(\theta)^2\Big)^{-1}\Big]^{\lambda}{}_{\rho}\,,\\
\tilde A_{3\mathrm{b}} &= -\frac{1}{6}\omega\theta^{\mu\nu\rho}i_\mu i_\nu i_\rho dx^0\wedge\cdots\wedge dx^5\,.
\end{aligned}
\tag{3.2}
$$

Here $\theta^{\mu\nu\rho}$ is a constant dimensionless anti-symmetric tensor, and $(\theta^2)^\mu{}_\nu$ is defined as follows:

$$
(\theta^2)^\mu{}_\nu = \theta^{\mu\nu'\rho}g_{\nu'\sigma}g_{\rho\sigma'}\theta^{\sigma\sigma'\lambda}g_{\lambda\nu}\,. \tag{3.3}
$$

The deformation parameter $\theta^{\mu\nu\rho}$ is constrained to only have 'one' non-zero component, e.g., $\theta^{\alpha\beta\gamma} = \theta\epsilon^{\alpha\beta\gamma}$ ($\epsilon^{012} = 1$), where $\alpha = 0,1,2$, while $\theta^{abc} = 0$, $a = 3,4,5$. It has been shown in [25, 14, 28] that one parameter is enough to parameterize all deformations of an M5-brane with a non-linearly self-dual three form A (up to Lorentz transformations). It is therefore no restriction to constrain $\theta^{\mu\nu\rho}$ to only have one parameter. An important difference between the M5-brane and D4-brane is that from an M5-brane point of view the difference between rank 2 and rank 4 B-field on the D4-brane is a Lorentz transformation (see e.g., [30]).

This solution generating technique works similarly to the $O(p+1,p+1)$ method. This means that if we, e.g., want to turn on a non-zero magnetic three form, we start with an undeformed M5-brane which we compactify on a three-torus. Next, we invert the volume of the torus, which implies that the M5-brane becomes an M2-brane smeared in three

[4]Note that this M5-brane solution is assumed to reduce to (2.2), with $p = 4$.

directions. This is followed by a gauge transformation of the three form A in the three directions which the M2-brane is smeared, e.g., $A_{345} = 0 \to A_{345} = \theta$. Finally, we invert the volume of the three-torus and then we decompactify. This interpretation of the solution generating technique will be further motivated in section 6.

Next, we are going to give the following arguments why (3.2) is correct.

1. We show that a double dimensional reduction along the y direction to a rank 2 $\theta_{\rm s}^{\mu\nu} = \theta^{\mu\nu y}$, gives (2.4) constrained to rank 2.

2. We will show that (3.2) generates the correct half-supersymmetric M5-M2 and M5-M2-M2-MW bound states. This further implies that (3.2) gives the correct solution if we start with an undeformed M5-brane solution with a conformally flat metric in the M5-brane directions, see section 3.2.

3. We will also show that (3.2) implies that the open membrane metric is manifestly deformation independent under electric or light-like deformations, while the generalized noncommutativity parameter is constant as expected. We note that since an electric and a magnetic deformation must be related through a coordinate transformation, the open membrane metric is obviously *not* deformation independent under a magnetic deformation. However, the generalized noncommutativity parameter is still constant. The reason that deformation independence is important to show is because it was used in the construction of the open membrane metric in [28]. However, there deformation independence was *assumed* for *one* particular deformed solution. Here we show that any electric (and light-like) deformation gives manifestly deformation independent solutions, see section 3.3.

4. Further, in section 3.4 we show that (3.2) satisfies the non-linear self-duality equation for the three form A (in the M5-brane directions). We further show that $*\tilde{H}_4 = \tilde{H}_7$ and most importantly that the eleven-dimensional supergravity 'tensor' equation of motion for the three form A is satisfied.

5. We are also going to show that a reduction to type IIA with $\theta^{\mu\nu\rho}$, gives a new formula for one parameter RR three form deformations of D4-branes. This new formula is shown to give all the expected results, see section 4.2. For a relation to a formula for one parameter RR three form deformations of NS5-branes, see section 5.2.

6. Finally, we show that for a magnetic deformation, the deformed metric and three form (in the deformed directions) can be obtained by considering certain $\mathrm{SL}(2,\mathbb{R})$ transformations of the Kähler structure parameter for the three-torus, on which the M5-brane is compactified, see section 6.

Together, these tests strongly indicate that the solution generating technique (3.2) is correct. However, they are not enough to rigorously prove that (3.2) is correct. A rigorous proof would be to also show that the eleven-dimensional 'Einsteins' equation of motion is satisfied by (3.2). This, however, seems to be very difficult to show, because 'Einsteins' equation of motion is second order in derivatives of the deformed metric. This is an important difference (complication) compared to the 'tensor' equation, which avoids explicit derivatives on the deformed metric, see section 3.4. A complete proof of (3.2) could also be obtained if there existed a microscopic formulation of M-theory 'T-duality'. Then (3.2) could be derived from the M-theory 'T-duality' rules. In a future paper [33] we plan to further investigate the latter possibility.

3.2 M5-M2 and M5-M2-M2-MW solutions

In this subsection we will show that using (3.2) gives the correct half-supersymmetric supergravity solutions corresponding to M5-M2 and M5-M2-M2-MW bound states.

We begin by giving the half-supersymmetric M5-brane solution [34]:

$$\begin{aligned} ds^2 &= H^{-1/3}\eta_{\mu\nu}dx^\mu dx^\nu + H^{2/3}\delta_{mn}dx^m dx^n\,, \qquad \mu,\nu = 0,1,\ldots,5\,, \quad m,n = 6,7,8,9,10\,,\\ A_6 &= H^{-1}dx^0\wedge\cdots\wedge dx^5\,, \quad A_3 = \gamma_3\,, \qquad H = 1+\frac{R^3}{r^3}\,, \end{aligned} \tag{3.4}$$

where H is a harmonic function on the transverse space and γ_3 is the three form dual to the six form, i.e., $\gamma = 3R^3\epsilon_3$, where $d\epsilon_3$ is the volume form of the four-sphere. Now, continuing by deforming this M5-brane solution with $\theta^{012} = \theta$, using (3.2), gives[5]

$$\begin{aligned} d\tilde{s}^2 &= (Hh_-^{-1})^{-1/3}\Big[\frac{1}{h_-}\Big((dx^0)^2+(dx^1)^2+(dx^2)^2\Big)+(dx^3)^2+(dx^4)^2+(dx^5)^2\Big]+\\ &\quad + h_-^{1/3}H^{2/3}\delta_{mn}dx^m dx^n\,,\\ \tilde{A}_3 &= \frac{\theta}{Hh_-}dx^0\wedge dx^1\wedge dx^2 + \frac{\theta}{H}dx^3\wedge dx^4\wedge dx^5 + \gamma_3\,,\\ \tilde{A}_6 &= (Hh_-)^{-1}\Big(\frac{h_-+1}{2}\Big)dx^0\wedge\cdots\wedge dx^5 +\\ &\quad +\frac{1}{2}\theta H^{-1}(h_-^{-1}dx^0\wedge dx^1\wedge dx^2 + dx^3\wedge dx^4\wedge dx^5)\wedge\gamma_3\,,\\ h_- &= 1-\theta^2H^{-1}\,, \end{aligned} \tag{3.5}$$

while a deformation with $\theta^{345} = -\theta'$, gives

$$\begin{aligned} d\tilde{s}^2 &= (Hh_+^{-1})^{-1/3}\Big[(dx^0)^2+(dx^1)^2+(dx^2)^2+\frac{1}{h_+}\Big((dx^3)^2+(dx^4)^2+(dx^5)^2\Big)\Big]+\\ &\quad + h_+^{1/3}H^{2/3}\delta_{mn}dx^m dx^n\,,\\ \tilde{A}_3 &= \frac{\theta'}{H}dx^0\wedge dx^1\wedge dx^2 + \frac{\theta'}{Hh_+}dx^3\wedge dx^4\wedge dx^5 + \gamma_3\,,\\ \tilde{A}_6 &= (Hh_+)^{-1}\Big(\frac{h_++1}{2}\Big)dx^0\wedge\cdots\wedge dx^5 +\\ &\quad +\frac{1}{2}\theta' H^{-1}(dx^0\wedge dx^1\wedge dx^2 + h_+^{-1}dx^3\wedge dx^4\wedge dx^5)\wedge\gamma_3\,,\\ h_+ &= 1+\theta'^2H^{-1}\,. \end{aligned} \tag{3.6}$$

[5] As far as we know this is the first time the M5-M2 brane solution has been given with both the three form and the dual six form. We note that the M5-M2 bound state was first obtained in [35]. Note that their solution is written in a different form than ours.

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Finally, deforming (3.4) with a light-like $\theta^{-12} = \theta$ (where $x^{\pm} = \frac{1}{\sqrt{2}}(x^5 \pm x^0)$), gives

$$
\begin{aligned}
d\tilde{s}^2 &= H^{-1/3}(2dx^- dx^+ - \theta^2 H^{-1}(dx^+)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2) + \\
&\quad + H^{2/3}\delta_{mn} dx^m dx^n \,, \\
\tilde{A}_6 &= H^{-1} dx^0 \wedge \cdots \wedge dx^5 - \\
&\quad - \frac{1}{2}\theta H^{-1} dx^+ \wedge (dx^1 \wedge dx^2 + dx^3 \wedge dx^4) \wedge \gamma_3 \,, \\
\tilde{A}_3 &= -\frac{\theta}{H} dx^+ \wedge (dx^1 \wedge dx^2 + dx^3 \wedge dx^4) + \gamma_3 \,.
\end{aligned} \tag{3.7}
$$

Note that if we instead deform with $\theta^{-34} = -\theta$, we obtain the same result as in (3.7).

Of these three solutions, the first two correspond to equivalent M5-M2 bound states,[6] while the third corresponds to an M5-M2-M2-MW bound state with equal absolute value of the M2-brane charges.

3.3 Open membrane data and deformation independence

Here we show that (3.2) inserted in the open membrane metric and generalized noncommutativity (theta) parameter, gives the expected results, i.e., for an electric or a light-like deformation $\tilde{G}^{\rm OM}_{\mu\nu}$ is independent of the deformation parameter θ, while $\Theta^{\mu\nu\rho}_{\rm OM}$ is constant. The open membrane metric and generalized noncommutativity parameter are given by [28]–[30]:[7]

$$
\begin{aligned}
\tilde{G}^{\rm OM}_{\mu\nu} &= \Big(\frac{1-\sqrt{1-K^{-2}}}{K^2}\Big)^{1/3}\Big(\tilde{g}_{\mu\nu} + \frac{1}{4}(\tilde{A}^2)_{\mu\nu}\Big) \,, \\
\Theta^{\mu\nu\rho}_{\rm OM} &= -\ell_{\rm p}^3 [K(1-\sqrt{1-K^{-2}})]^{2/3} \tilde{g}^{\mu\mu_1} \tilde{A}_{\mu_1\nu_1\rho_1} \tilde{G}^{\nu_1\nu}_{\rm OM} \tilde{G}^{\rho_1\rho}_{\rm OM} \,,
\end{aligned} \tag{3.8}
$$

where

$$
(\tilde{A}^2)_{\mu\nu} = \tilde{g}^{\mu_1\nu_1}\tilde{g}^{\mu_2\nu_2}\tilde{A}_{\mu_1\mu_2\mu}\tilde{A}_{\nu_1\nu_2\nu} \,, \qquad \tilde{A}^2 = \tilde{g}^{\mu\nu}(\tilde{A}^2)_{\mu\nu} \,, \qquad K = \sqrt{1 + \frac{1}{24}\tilde{A}^2} \,. \tag{3.9}
$$

To simplify the calculations (for the electric and magnetic cases) we use that if one goes to a frame (u^α_μ, v^a_μ), where $\alpha = 0,1,2$ and $a = 3,4,5$, parameterize the coset $\mathrm{SO}(1,5)/\,\mathrm{SO}(1,2)\times \mathrm{SO}(3)$ as defined in [14], then the open membrane metric and theta parameter can be written as:

$$
\begin{aligned}
\tilde{G}^{\rm OM}_{\alpha\beta} &= \Big(1 + \frac{1}{6}\tilde{A}_1^2\Big)^{2/3}\tilde{g}_{\alpha\beta} \,, \qquad \tilde{A}_1^2 = \tilde{g}^{\alpha\beta}(\tilde{A}_1^2)_{\alpha\beta} \,, \\
(\tilde{A}_1^2)_{\alpha\beta} &= \tilde{g}^{\alpha_1\beta_1}\tilde{g}^{\alpha_2\beta_2}\tilde{A}_{\alpha_1\alpha_2\alpha}\tilde{A}_{\beta_1\beta_2\beta} \,, \qquad \alpha,\beta = 0,1,2 \,, \\
\tilde{G}^{\rm OM}_{ab} &= \Big(1 + \frac{1}{6}\tilde{A}_2^2\Big)^{1/3}\tilde{g}_{ab} \,, \qquad \tilde{A}_2^2 = \tilde{g}^{ab}(\tilde{A}_2^2)_{ab} \,, \\
(\tilde{A}_2^2)_{ab} &= \tilde{g}^{a_1b_1}\tilde{g}^{a_2b_2}\tilde{A}_{a_1a_2a}\tilde{A}_{b_1b_2b} \,, \qquad a,b = 3,4,5 \,,
\end{aligned} \tag{3.10}
$$

[6]Further, after taking appropriate near horizon limits (electric and magnetic, respectively, see [18]) of (3.5) and (3.6) we obtain solutions which are not only equivalent but identical. This solution is the supergravity dual of OM-theory [13, 16].

[7]For related work concerning a three index structure and open membranes, see [36].

and

$$\Theta_{\rm OM}^{\alpha\beta\gamma} = -\ell_{\rm p}^3(1+\frac{1}{6}\tilde{A}_2^2)\tilde{A}^{\alpha\beta\gamma}\ , \quad \Theta_{\rm OM}^{abc} = -\ell_{\rm p}^3(1+\frac{1}{6}\tilde{A}_1^2)\tilde{A}^{abc}\,. \tag{3.11}$$

To obtain these relations we have used that

$$\begin{aligned} K^2 &= \frac{(1+(1/12)\tilde{A}_i^2)^2}{1+(1/6)\tilde{A}_i^2}\,, \qquad i=1,2\,, \\ 1+\frac{1}{6}\tilde{A}_2^2 &= \Big(1+\frac{1}{6}\tilde{A}_1^2\Big)^{-1}\,, \end{aligned} \tag{3.12}$$

in the above parameterization. Note that equation (3.11) implies that $\Theta_{\rm OM}^{\mu\nu\rho}$ is completely anti-symmetric. In [28] it is further shown, using (3.11) and (3.10), that

$$*_{\tilde{G}}\,\Theta_{\rm OM} = \Theta_{\rm OM}\ , \quad (*_{\tilde{G}}\Theta_{\rm OM})^{\mu\nu\rho} = \frac{1}{6}\frac{1}{\sqrt{-\tilde{G}}}\epsilon^{\mu\nu\rho\iota\kappa\lambda}\Theta^{\rm OM}_{\iota\kappa\lambda}\,, \tag{3.13}$$

where $\tilde{G}$ is the determinant of the open membrane metric and the indices on $\Theta_{\rm OM}$ are lowered with $\tilde{G}^{\rm OM}_{\mu\nu}$. From this relation we see that $\Theta_{\rm OM}^{\mu\nu\rho}$ is *linearly* self-dual with respect to the open membrane metric $\tilde{G}^{\rm OM}_{\mu\nu}$.

Next, we continue by simplifying (3.2) in the above parameterization. For an electric deformation, i.e., $\theta^{012}=\theta$, the metric and three form in (3.2), in the M5-brane directions, can be written as

$$\begin{aligned} \tilde{g}_{\alpha\beta} &= \Big(1+\frac{1}{6}(\theta)^2\Big)^{-2/3}g_{\alpha\beta}\,, \qquad \tilde{g}_{ab} = \Big(1+\frac{1}{6}(\theta)^2\Big)^{1/3}g_{ab}\,, \\ \tilde{A}_{012} &= -\theta g_{(\alpha\beta)}\Big(1+\frac{1}{6}(\theta)^2\Big)^{-1}\,, \qquad \tilde{A}_{345} = \theta\omega = \theta\sqrt{-g_{(\mu\nu)}}\,, \end{aligned} \tag{3.14}$$

where $(\theta)^2 = g^{\alpha\beta}(\theta^2)_{\alpha\beta}$, $g_{(\alpha\beta)}$ is the determinant of $g_{\alpha\beta}$ and we have used that $(\theta^2)_{\alpha\beta} = \frac{1}{3}g_{\alpha\beta}(\theta)^2$, while $A_1^2=(\theta)^2$. For the purpose of later sections we note that

$$1+\frac{1}{6}(\theta)^2 = 1+\theta^2 g_{(\alpha\beta)}\,. \tag{3.15}$$

Inserting (3.14) in the open membrane metric and theta parameter gives

$$\tilde{G}^{\rm OM}_{\mu\nu} = g_{\mu\nu}\,, \qquad \Theta^{012}_{\rm OM} = \ell_{\rm p}^3\theta\,, \qquad \Theta^{345}_{\rm OM} = -\ell_{\rm p}^3\theta\sqrt{\frac{-g_{(\alpha\beta)}}{g_{(ab)}}}\,. \tag{3.16}$$

We see here that the open membrane metric is deformation independent and in the case when $g_{\mu\nu}$ is conformally flat, the theta parameter is $\Theta^{012}_{\rm OM} = \Theta^{345}_{\rm OM} = \ell_{\rm p}^3\theta$, which is the result we expected.

Next, if we repeat the above analysis for a magnetic deformation $\theta^{345}=-\theta$, we obtain the following result:

$$\tilde{G}^{\rm OM}_{\mu\nu} = \Big(1+\frac{1}{6}(\theta)^2\Big)^{-1/3}g_{\mu\nu}\,, \qquad \Theta^{345}_{\rm OM} = -\ell_{\rm p}^3\theta\,, \qquad \Theta^{012}_{\rm OM} = \ell_{\rm p}^3\theta\sqrt{\frac{g_{(ab)}}{-g_{(\alpha\beta)}}}\,. \tag{3.17}$$

Here the open membrane metric is not deformation independent, which we also expected, see [28]. Further, since electric and magnetic deformations give equivalent M5-M2 bound states, it is clear that $\sqrt{\frac{g_{(ab)}}{-g_{(\alpha\beta)}}}$ = constant, in both (3.16) and (3.17). This in turn implies that the theta parameter is always constant.

Next, we check the light-like case. For example, turning on θ^{-12}, where $x^{\pm} = \frac{1}{\sqrt{2}}(x^5 \pm x^0)$, gives the following open membrane data (Note that in this case (3.10) and (3.11) cannot be used):

$$\tilde{G}^{\mathrm{OM}}_{\mu\nu} = g_{\mu\nu}\,, \qquad \Theta^{-12}_{\mathrm{OM}} = \ell_{\mathrm{p}}^3\theta\,, \qquad \Theta^{-34}_{\mathrm{OM}} = \ell_{\mathrm{p}}^3\theta(g^{33}g^{44}g_{11}g_{22})^{1/2}\,. \tag{3.18}$$

Note that if $g_{\mu\nu}$ is conformally flat $g^{33}g^{44}g_{11}g_{22} = 1$ and $\Theta^{-12}_{\mathrm{OM}} = \Theta^{-34}_{\mathrm{OM}}$, see (3.7) for an example of this kind.

3.4 Non-linear self-duality and the 'tensor' equation of motion

Next, we continue by showing that the three form obtained from (3.2) obeys the non-linear self-duality equation in the M5-brane directions. The non-linear self-duality equation on the M5-brane can be written as [29]:

$$\begin{aligned} z^{-1}\tilde{G}^{\mu\sigma}_{\mathrm{OM}}\tilde{A}_{\sigma}{}^{\nu\rho} &= \frac{1}{6}\frac{1}{\sqrt{-\tilde{g}_{(\mu\nu)}}}\epsilon^{\mu\nu\rho\mu'\nu'\rho'}\tilde{A}_{\mu'\nu'\rho'}\,, \\ z^{-1} &= [K(1-\sqrt{1-K^{-2}})]^{1/3}\,, \end{aligned} \tag{3.19}$$

where $\epsilon^{012345} = -1$. We now check that (3.14) obeys the non-linear self-duality equation (3.19). Inserting (3.14) in the right hand side (RHS) and the LHS of (3.19), gives that RHS=LHS, i.e., the non-linear self-duality equation is satisfied. Repeating the calculation for a magnetic deformation also gives that (3.19) is satisfied. Note that for the non-linear self-duality to be satisfied $\omega = \sqrt{-g_{(\mu\nu)}}$ in (3.1), i.e., the zero force condition has to be satisfied by the undeformed solution. That the non-linear self-dulity equation is satisfied implies that electric and magnetic deformations give equivalent M5-M2 bound states[8].

Inserting a light-like deformation also gives that (3.19) is satisfied. In this case there is a linear self-duality equation.

We note that showing that (3.19) is satisfied is equivalent to showing that the *linear* self-duality equation (3.13) for $\Theta^{\mu\nu\rho}_{\mathrm{OM}}$ is satisfied. Using (3.16), (3.17) and (3.18), we obtain that the theta parameters, obtained in the last subsection, are linearly self-dual with respect to the open membrane metric.

Next, we show that (3.2) satisfies $*\tilde{H}_4 = \tilde{H}_7$ (with $\epsilon^{01\ldots 9,10} = -1$) under the restriction that all functions only depend on the transverse coordinates. Here

$$\tilde{H}_4 = d\tilde{A}_3\,, \qquad \tilde{H}_7 = d\tilde{A}_6 - \frac{1}{2}\tilde{A}_3 \wedge d\tilde{A}_3\,. \tag{3.20}$$

For (3.2) we obtain

$$\tilde{H}_7 = dA_6 - \tilde{A}_{3\mathrm{a}} \wedge d\tilde{A}_{3\mathrm{b}} - (\tilde{A}_{3\mathrm{a}} + \tilde{A}_{3\mathrm{b}}) \wedge d\gamma_3\,. \tag{3.21}$$

[8]This is in agreement with results obtained in [18], for deformations of D3-branes (see section 5 and in [18, appendix B]). There it is shown that a magnetic deformation of a D3-brane is S-dual to an electric deformation of a D3-brane, if the undeformed D3-brane solution satisfies the zero force condition.

For a one parameter deformation the relation $*\tilde{H}_4 = \tilde{H}_7$ gives three independent relations

$$(*_{\tilde{g}})d\gamma_3 = dA_6 - \tilde{A}_{3\mathrm{a}} \wedge d\tilde{A}_{3\mathrm{b}} \,, \qquad (*_{\tilde{g}})d\tilde{A}_{3\mathrm{a}} = -\tilde{A}_{3\mathrm{b}} \wedge d\gamma_3 \,, \qquad (*_{\tilde{g}})d\tilde{A}_{3\mathrm{b}} = -\tilde{A}_{3\mathrm{a}} \wedge d\gamma_3 \,. \tag{3.22}$$

We start by showing that the first relation in (3.22) is correct. For an electric or a magnetic deformation we obtain that the left hand side (LHS) is

$$(*_{\tilde{g}})d\gamma_3 = h^{-1}[(*_g)d\gamma_3] \,, \qquad h = 1 + \theta^2 g_{(\alpha\beta)} \,, \tag{3.23}$$

while the right hand side (RHS) is

$$dA_6 - \tilde{A}_{3\mathrm{a}} \wedge d\tilde{A}_{3\mathrm{b}} = h^{-1} dA_6 \,, \tag{3.24}$$

where $\alpha, \beta = 0, 1, 2$ for an electric deformation while $\alpha, \beta = 3, 4, 5$ for a magnetic deformation. The result obtained in (3.23) and (3.24) implies that if the LHS should be equal to the RHS then $(*g)d\gamma_3 = dA_6$. Since precisely this relation was demanded for the undeformed solution (3.1), we have therefore shown that the first relation in (3.22) is correct. For a light-like deformation the first relation in (3.22) is trivially satisfied. Next, using that $(*_{\tilde{g}})dA_{3(\mathrm{b})} = h^{-1}[(*_g)dA_{3(\mathrm{b})}]$, we obtain that the third relation in (3.22) is satisfied for all kinds of deformations. Further, since electric and magnetic deformations give equivalent solutions, the second relation must be correct due to the fact that the third relation is correct.

Next, we use that $*\tilde{H}_4 = \tilde{H}_7$ in order to show that the eleven-dimensional supergravity 'tensor' equation of motion

$$d(*_{\tilde{g}}\tilde{H}_4) = -\frac{1}{2}\tilde{H}_4 \wedge \tilde{H}_4 \,, \tag{3.25}$$

is satisfied. Using that $*\tilde{H}_4 = \tilde{H}_7$ and (3.20), we obtain that the LHS of (3.25) is given by

$$\mathrm{LHS} = -d\tilde{A}_{3\mathrm{a}} \wedge d\tilde{A}_{3\mathrm{b}} - (d\tilde{A}_{3\mathrm{a}} + d\tilde{A}_{3\mathrm{b}}) \wedge d\gamma_3 \,, \tag{3.26}$$

while using that $\tilde{H}_4 = d\tilde{A}_{3\mathrm{a}} + d\tilde{A}_{3\mathrm{b}} + d\gamma_3$, we obtain that the RHS of (3.25) is given by

$$\mathrm{RHS} = -d\tilde{A}_{3\mathrm{a}} \wedge d\tilde{A}_{3\mathrm{b}} - (d\tilde{A}_{3\mathrm{a}} + d\tilde{A}_{3\mathrm{b}}) \wedge d\gamma_3 \,. \tag{3.27}$$

Comparing (3.26) and (3.27) we find that (3.25) is satisfied.

4. Reduction to type IIA string theory

In this section we are going to show that a double dimensional reduction of (3.2) for a one parameter deformation leads to the correct type IIA expression for a rank 2 NS-NS two form deformation of a D4-brane. Reducing longitudinal to the M5-brane but transverse to the deformation directions, leads to a new formula for one parameter three form RR deformations of D4-branes.

4.1 Reduction to rank 2 B-field

Next, we continue by showing that (3.2) gives (2.4) (for $p = 4$) with rank 2 $\theta_{\rm s}^{\mu\nu}$ under double dimensional reduction of $\theta^{\mu\nu\rho}$. We use the following relations between eleven and ten-dimensional fields (under the restriction $g_{\mu y} = 0$):

$$\begin{aligned}
\frac{g_{MN}}{\ell_{\rm p}^2} &= e^{-2\phi/3}\,\frac{g^{(\rm s)}_{MN}}{\alpha'} = \frac{g^{\rm D2}_{MN}}{\alpha'}\,, \qquad \frac{g_{yy}}{\ell_{\rm p}^2} = \frac{e^{\frac{4\phi}{3}}}{R^2}\,,\\
\frac{A_3}{\ell_{\rm p}^3} &= \frac{C_3}{\alpha'^{\frac{3}{2}}} + \frac{B_2}{\alpha'} \wedge \frac{dy}{R}\,,\\
\frac{A_6}{\ell_{\rm p}^6} &= \frac{B_6}{\alpha'^3} + \Big(\frac{C_5}{\alpha'^{\frac{5}{2}}} + \frac{1}{2}\frac{C_3}{\alpha'^{\frac{3}{2}}} \wedge \frac{B_2}{\alpha'}\Big) \wedge \frac{dy}{R}\,,
\end{aligned} \tag{4.1}$$

where $M, N = 0, 1, \ldots, 9$, R is the radius of the compactified direction labeled by y and $g_{\mu y} = 0$. We also use the following standard parameter relations $\ell_{\rm p}^2 = g^{2/3}\alpha'$ and $R = g\sqrt{\alpha'}$.

We start by setting $\theta_{\rm s}^{\alpha\beta} = \theta^{\alpha\beta y}$, where $\alpha, \beta = 0, 1, \text{or } 3, 4$, i.e., electric or magnetic, while a, b are the other three directions and y is the direction in which we reduce. This implies that

$$\frac{1}{2}(\theta^2)^{\alpha}{}_{\beta} = -(\theta_{\rm s}^2)^{\alpha}{}_{\beta}\,, \qquad (\theta)^2 = -3(\theta_{\rm s})^2\,, \qquad \Big[\Big(1 + \frac{1}{2}(\theta)^2\Big)^{-1}\Big]^{y}{}_{y} = [\det(1 - (\theta_{\rm s})^2)]^{-1/2}\,. \tag{4.2}$$

Further, we obtain that

$$\det(1 - (\theta_{\rm s})^2) = \Big[\det\Big(1 + \frac{1}{2}(\theta)^2\Big)\Big]^{2/3}\,. \tag{4.3}$$

Here we have used (4.2) and that

$$\det\Big(1 + \frac{1}{2}(\theta)^2\Big) = \Big(1 + \frac{1}{6}(\theta)^2\Big)^3\,, \qquad \det(1 - (\theta_{\rm s})^2) = \Big(1 - \frac{1}{2}(\theta_{\rm s})^2\Big)^2\,, \tag{4.4}$$

where we have used that $(\theta_{\rm s}^2)_{\alpha\beta} = (\theta_{\rm s})^2 \frac{1}{2} g^{(\rm s)}_{\alpha\beta}$.

We now use (4.1)-(4.4) in (3.2), which gives (2.4) with $p = 4$ and that $\theta_{\rm s}$ is rank 2 (electric or magnetic), i.e., a one parameter deformation. However, as has been shown in [22], the formula (2.4) is valid also for a rank 4 deformation. The rank 4 case should be possible to obtain from a skew reduction of (3.2).

The above calculations confirm that (3.2) under a double dimensional reduction of the three index theta, gives (2.4) with $p = 4$ and rank 2 $\theta_{\rm s}$.

4.2 Reduction to one parameter RR three form deformation of a D4-brane

In this subsection, we again reduce (3.2) to ten dimensions. But instead of reducing the three index theta to a two index theta we will reduce to a three index theta, i.e., reduce in a direction 'transverse' to the deformation. This implies that we obtain a formula for deforming a D4-brane with a one parameter RR three form. Note that the undeformed solution is given in (2.2) with $p = 4$. The reduction is straight forward and using (4.1) we

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obtain (with $\mu,\nu=0,1,\ldots,4$)

$$
\begin{aligned}
\tilde{g}^{(\mathrm{s})}_{\mu\nu} &= \Big[\det\Big(1+\frac{1}{2}(\theta)^2\Big)\Big]^{1/6} g^{(\mathrm{s})}_{\mu\rho}\Big[\Big(1+\frac{1}{2}(\theta)^2\Big)^{-1}\Big]^{\rho}{}_{\nu}\,, \quad \tilde{g}^{(\mathrm{s})}_{mn} = \Big[\det\Big(1+\frac{1}{2}(\theta)^2\Big)\Big]^{1/6} g^{(\mathrm{s})}_{mn}\,,\\
g\tilde{C}_3 &= g\tilde{C}_{3\mathrm{a}}+\gamma_3\,, \qquad e^{2\tilde{\phi}} = e^{2\phi}\Big[\det\Big(1+\frac{1}{2}(\theta)^2\Big)\Big]^{1/6}\,, \qquad g^2\tilde{B}_6 = \frac{1}{2}g\tilde{C}_{3\mathrm{a}}\wedge\gamma_3\,,\\
g\tilde{C}_5 &= \omega dx^0\wedge\ldots\wedge dx^4 - \tilde{B}_2\wedge\gamma_3\,, \qquad \tilde{B}_2 = -\omega\frac{1}{6}\theta^{\mu\nu\rho}i_\mu i_\nu i_\rho dx^0\wedge\cdots\wedge dx^4\,,
\end{aligned}
$$

where

$$
g\tilde{C}_{3\mathrm{a}} = \frac{1}{6}g\tilde{C}^{3\mathrm{a}}_{\mu\nu\rho}dx^\mu\wedge dx^\nu\wedge dx^\rho\,, \quad g\tilde{C}_{\mu\nu\rho} = -g^2e^{-2\phi}g^{(\mathrm{s})}_{\mu\rho'}g^{(\mathrm{s})}_{\sigma\nu}\theta^{\rho'\sigma\sigma'}g^{(\mathrm{s})}_{\sigma'\lambda}\Big[\Big(1+\frac{1}{2}(\theta)^2\Big)^{-1}\Big]^{\lambda}{}_{\rho}\,. \tag{4.5}
$$

Here $\theta^{\mu\nu\rho}$ is a dimensionless (one parameter) anti-symmetric tensor, and $(\theta^2)^\mu{}_\nu$ is defined as follows:

$$
(\theta^2)^\mu{}_\nu = g^2e^{-2\phi}\theta^{\mu\nu'\rho}g^{(\mathrm{s})}_{\nu'\sigma}g^{(\mathrm{s})}_{\rho\sigma'}\theta^{\sigma\sigma'\lambda}g^{(\mathrm{s})}_{\lambda\nu} = g^2\theta^{\mu\nu'\rho}g^{\mathrm{D2}}_{\nu'\sigma}g^{\mathrm{D2}}_{\rho\sigma'}\theta^{\sigma\sigma'\lambda}g^{\mathrm{D2}}_{\lambda\nu}\,. \tag{4.6}
$$

We note that since this formula has been obtained from a direct dimensional reduction of a one parameter formula, it is only valid for one parameter deformations. For example, deforming with both $\theta^{012}\neq 0$ and $\theta^{234}\neq 0$ is not possible using (4.5) since the non-zero RR one form would be missing.

The above obtained formula (4.5) can be used to deform D4-branes by turning on a non-zero RR three form. For the half-supersymmetric case we have checked that electric, magnetic and light-like deformations give the correct solutions corresponding to D4-D2, D4-F1 and D4-D2-F1-W bound states, respectively. These solutions in the particular form obtained here have been obtained before in [37, 26], using other methods.[9] As an extra check that (4.5) is correct we have checked that T-dualizing in a direction parallel to the deformation followed by S-duality, gives (2.4) with $p=3$, as expected.

Next, we derive that the above formula implies that the open D2-brane metric and generalized noncommutativity parameter are manifestly deformation independent under one parameter deformations. The open D2-brane metric and generalized noncommutativity parameter are given by [28]:

$$
\tilde{G}^{\widetilde{\mathrm{OD2}}}_{\mu\nu} = \Big[1+\frac{1}{6}\tilde{C}_3^2\Big]^{-1/3}\left(\tilde{g}^{\mathrm{D2}}_{\mu\nu}+\frac{1}{2}(\tilde{C}_3^2)_{\mu\nu}\right), \tag{4.7}
$$

$$
\Theta^{\mu_1\mu_2\mu_3}_{\widetilde{\mathrm{OD2}}} = -(\alpha')^{3/2}\left(1+\frac{1}{6}\tilde{C}_3^2\right)^{1/3}\tilde{g}^{\mu_1\nu_1}_{\mathrm{D2}}\tilde{C}_{\nu_1\nu_2\nu_3}\tilde{G}^{\nu_2\mu_2}_{\mathrm{OD2}}\tilde{G}^{\nu_3\mu_3}_{\mathrm{OD2}}\,, \tag{4.8}
$$

where $\tilde{g}^{\mathrm{D2}}_{\mu\nu} = e^{-2\tilde{\phi}/3}\tilde{g}^{(\mathrm{s})}_{\mu\nu}$ is the closed (deformed) D2-brane metric, and

$$
(\tilde{C}_3^2)_{\mu\nu} = \tilde{g}^{\rho_1\sigma_1}_{\mathrm{D2}}\tilde{g}^{\rho_2\sigma_2}_{\mathrm{D2}}\tilde{C}_{\rho_1\rho_2\mu}\tilde{C}_{\sigma_1\sigma_2\nu}\,, \qquad \tilde{C}_3^2 = \tilde{g}^{\mu\nu}_{\mathrm{D2}}(\tilde{C}_3^2)_{\mu\nu}\,. \tag{4.9}
$$

[9] In [37] it was shown that deforming a half-supersymmetric D4-brane with, e.g., a rank 2 magnetic B-field is equivalent to deforming a half-supersymmetric D4-brane with an electric RR three form.

Using (4.5) and that $\theta^{012} = \theta$ *or* $\theta^{234} = \theta$, i.e., we have electric or magnetic deformation, gives

$$\begin{aligned} \tilde{g}^{\rm D2}_{\alpha\beta} &= \Big(1 + \frac{1}{6}(\theta)^2\Big)^{-2/3} g^{\rm D2}_{\alpha\beta}\,, \qquad \tilde{g}^{\rm D2}_{ab} = \Big(1 + \frac{1}{6}(\theta)^2\Big)^{1/3} g^{\rm D2}_{ab}\,, \\ \frac{1}{2}(\tilde{C}^2_3)_{\alpha\beta} &= \frac{1}{6}(\theta)^2\Big(1 + \frac{1}{6}(\theta)^2\Big)^{-2/3} g^{\rm D2}_{\alpha\beta}\,, \qquad \tilde{C}^2_3 = (\theta)^2\,, \end{aligned} \tag{4.10}$$

where $\alpha, \beta = 0, 1, 2$, or $2, 3, 4$, while $a, b = 3, 4$, or $0, 1$, respectively, for electric and magnetic deformations. Next, using (4.10) in (4.7) and (4.8), gives the following *deformation independent* open D2-brane metric and generalized noncommutativity parameter:

$$\tilde{G}^{\rm D2}_{\mu\nu} = g^{\rm D2}_{\mu\nu}\,, \qquad \Theta^{\alpha\beta\gamma}_{\rm D2} = g\alpha'^{3/2}\theta\epsilon^{\alpha\beta\gamma}\,. \tag{4.11}$$

Here $\epsilon^{012} = \epsilon^{234} = 1$. Also for a light-like deformation we obtain deformation independence. It is important that we have obtained that the open D2-brane metric and generalized noncommutativity parameter are deformation independent, since this means that any one parameter deformation of any kind of D4-brane solution,[10] gives manifestly deformation independent open D2-brane metric and generalized noncommutativity parameter. In [28, 26] this was shown for a one parameter deformation of the half-supersymmetric D4-brane. The result here is more general.

5. Deformation of IIA/B NS5-branes with RR three or two forms

In this section we derive formulas for deforming type IIA NS5-branes with one parameter RR three or two forms, respectively. For the case with an NS5-brane with non-zero RR three form we also derive the open D2-brane coupling which, e.g., is relevant for the OD2-theory [13].

5.1 Deformation of NS5-branes with a RR two form

In this subsection we show how a type IIB NS5-brane can be deformed by turning on a non-zero rank 2 RR two form. We start with the following undeformed type IIB NS5-brane solution

$$\begin{aligned} ds^2 &= g^{(\rm s)}_{\mu\nu}dx^\mu dx^\nu + g^{(\rm s)}_{mn}dx^m dx^n\,, \qquad e^{2\phi} = g^2\hat{F}\,, \\ g^2 B_6 &= -\omega dx^0\wedge\cdots\wedge dx^5 \qquad B_2 = \gamma_2\,, \end{aligned} \tag{5.1}$$

where $\hat{F}$ is some function, $\omega = g^2 e^{-2\phi}\sqrt{-g^{(\rm s)}_{(\mu\nu)}}$ due to the zero force condition, g is the closed string coupling constant and x^μ, $\mu = 0, \ldots, 5$, are coordinates in the brane directions, while x^m, $m = 6, \ldots, 9$, are coordinates in the transverse directions. Note that the two form $B = \gamma_2$ is dual to the six form B_6. For example, for a maximally supersymmetric NS5-brane $B \sim \epsilon_2$, where $d\epsilon_2$ is the volume form of the three-sphere. The above solution (5.1) is assumed to be T-dual to the undeformed type IIA solution given below in (5.13), and S-dual to (2.2) with $p = 5$.

[10] Of course under the restriction that the solution generating technique is valid.

The easiest way to obtain the formula for a type IIB NS5-brane deformed by a rank 2 RR two form is to S-dualize the formula for a type IIB D5-brane deformed by a rank 2 B-field (see (2.4) with $p=5$). We will use the following conventions for S-duality (assuming zero axion and only rank 2 B-field)

$$\begin{aligned}
\frac{g^{\rm s}_{MN}}{\alpha'_{\rm s}} &= e^{-\phi}\frac{g_{MN}}{\alpha'}\,, \qquad e^{\phi_{\rm s}} = e^{-\phi}\,,\\
\frac{B^{\rm s}_2}{\alpha'_{\rm s}} &= \frac{C_2}{\alpha'}\,, \qquad \frac{C^{\rm s}_2}{\alpha'_{\rm s}} = -\frac{B_2}{\alpha'}\,, \qquad \frac{C^{\rm s}_4}{(\alpha'_{\rm s})^2} = \frac{C_4}{(\alpha')^2} + \frac{B_2}{\alpha'}\wedge\frac{C_2}{\alpha'}\,,\\
\frac{B^{\rm s}_6}{(\alpha'_{\rm s})^3} &= -\frac{C_6}{(\alpha')^3} - \frac{1}{2}\frac{B_2}{\alpha'}\wedge\frac{C_4}{(\alpha')^2}\,, \qquad \frac{C^{\rm s}_6}{(\alpha'_{\rm s})^3} = \frac{B_6}{(\alpha')^3} - \frac{1}{2}\frac{C_2}{\alpha'}\wedge\frac{C_4}{(\alpha')^2}\,,
\end{aligned} \tag{5.2}$$

where $\alpha'_{\rm s} = g\alpha'$ and the index s means the S-dualized quantity.

Next, we are going to S-dualize (2.4) with $p=5$ for a rank 2 NS-NS deformation, where (2.4) with $p=5$ restricted to rank 2 B-field is given by

$$\begin{aligned}
\tilde{g}^{(\rm s)}_{\mu\nu} &= g^{(\rm s)}_{\mu\rho}\left[\left(1-(\theta)^2\right)^{-1}\right]^{\rho}{}_{\nu}\,, \qquad \tilde{g}^{(\rm s)}_{mn} = g^{(\rm s)}_{mn}\,,\\
g'\tilde{C}_2 &= \gamma_2\,, \qquad \tilde{B}_2 = \tilde{B}_{2{\rm a}}\,, \qquad e^{2\tilde{\phi}} = e^{2\phi'}\left[\det\left(1-(\theta)^2\right)\right]^{-1/2}\,,\\
g'\tilde{C}_4 &= g'\tilde{C}_{4{\rm b}} - \tilde{B}_2\wedge\gamma_2\,, \qquad g'^2\tilde{B}_6 = \frac{1}{2}g'\tilde{C}_4\wedge\gamma_2\,,\\
\tilde{C}_6 &= C_6 - \tilde{B}_2\wedge\tilde{C}_{4{\rm b}}\,, \qquad g'\tilde{C}_{4{\rm b}} = \omega\frac{1}{2}\theta^{\mu\nu}i_\mu i_\nu dx^0\wedge\cdots\wedge dx^5\,,
\end{aligned} \tag{5.3}$$

where

$$\tilde{B}_{2{\rm a}} = \frac{1}{2}\tilde{B}^{2{\rm a}}_{\mu\nu}dx^\mu\wedge dx^\nu\,, \qquad \tilde{B}^{2{\rm a}}_{\mu\nu} = -g^{(\rm s)}_{\mu\rho}\theta^{\rho\sigma}g^{(\rm s)}_{\sigma\lambda}\left[\left(1-(\theta)^2\right)^{-1}\right]^{\lambda}{}_{\nu}\,, \tag{5.4}$$

and

$$(\theta^2)^{\mu}{}_{\nu} = \theta^{\mu\nu'}g^{(\rm s)}_{\nu'\rho}\theta^{\rho\sigma}g^{(\rm s)}_{\sigma\nu}\,. \tag{5.5}$$

Note that the formulas for the metric, dilaton and NS-NS two form would not change if we considered rank 4 and rank 6 deformations. Continuing by S-dualizing (5.4), using (5.2), we obtain the following formula for a type IIB NS5-brane deformed with a rank 2 RR two form (note that $B_6 = -C_6$)

$$\begin{aligned}
\tilde{g}^{(\rm s)}_{\mu\nu} &= \left[\det\left(1-(\theta)^2\right)\right]^{1/4}g^{(\rm s)}_{\mu\rho}\left[\left(1-(\theta)^2\right)^{-1}\right]^{\rho}{}_{\nu}\,, \quad \tilde{g}^{(\rm s)}_{mn} = \left[\det\left(1-(\theta)^2\right)\right]^{1/4}g^{(\rm s)}_{mn}\,,\\
\tilde{C}_2 &= \tilde{C}_{2{\rm a}}\,, \qquad \tilde{B}_2 = \gamma_2\,, \qquad e^{2\tilde{\phi}} = e^{2\phi}\left[\det\left(1-(\theta)^2\right)\right]^{1/2}\,, \qquad \tilde{C}_6 = 0\,,\\
\tilde{B}_6 &= B_6 - \frac{1}{2}g\tilde{C}_2\wedge g\tilde{C}_4\,, \qquad g\tilde{C}_4 = \omega\frac{1}{2}\theta^{\mu\nu}i_\mu i_\nu dx^0\wedge\cdots\wedge dx^5\,,
\end{aligned} \tag{5.6}$$

where

$$g\tilde{C}_{2{\rm a}} = \frac{1}{2}g\tilde{C}^{2{\rm a}}_{\mu\nu}dx^\mu\wedge dx^\nu\,, \qquad g\tilde{C}^{2{\rm a}}_{\mu\nu} = g^2e^{-2\phi}g^{(\rm s)}_{\mu\rho}\theta^{\rho\sigma}g^{(\rm s)}_{\sigma\lambda}\left[\left(1-(\theta)^2\right)^{-1}\right]^{\lambda}{}_{\nu}\,, \tag{5.7}$$

and

$$(\theta^2)^{\mu}{}_{\nu} = g^2e^{-2\phi}\theta^{\mu\nu'}g^{(\rm s)}_{\nu'\rho}\theta^{\rho\sigma}g^{(\rm s)}_{\sigma\nu} = g^2\theta^{\mu\nu'}g^{\rm D1}_{\nu'\rho}\theta^{\rho\sigma}g^{\rm D1}_{\sigma\nu}\,. \tag{5.8}$$

This formula is valid for an NS5-brane deformed with a rank 2 RR two form, except for the expressions for the metric and RR two form which are valid also for rank 4 and 6 deformations, while the expression for the dilaton is valid for a rank 4 but not rank 6 deformation.

Next, we show that the metric and RR two form, under a deformation of the NS5-brane, transform similarly to how the metric and the B-field transform under a deformation of a Dp-brane. We begin by introducing the following tensor

$$F_{\mu\nu} = g g^{\mathrm{D1}}_{\mu\nu} - g C_{\mu\nu}\,, \tag{5.9}$$

where $g^{\mathrm{D1}}_{\mu\nu} = e^{-\phi} g^{(\mathrm{s})}_{\mu\nu}$ and g is the closed (fundamental) string coupling constant. If we now perform the same projective transformation for $F_{\mu\nu}$ as we did for the tensor $E_{\mu\nu}$ in section 2, we get (starting with $C_{\mu\nu} = 0$)

$$\tilde{F}_{\mu\nu} = \Big(\frac{F}{\theta F + 1}\Big)_{\mu\nu} = \Big(\frac{g g^{\mathrm{D1}}(1 - \theta g g^{\mathrm{D1}})}{(1 + \theta g g^{\mathrm{D1}})(1 - \theta g g^{\mathrm{D1}})}\Big)_{\mu\nu}\,. \tag{5.10}$$

From (5.10) we easily obtain the closed D1-brane metric and RR two form $\tilde{C}_2$. If we compare with (5.7) we find that the RR two form is the same and computing the closed D1-brane metric, using the closed string metric and dilaton given in (5.6), we find the same answer as we obtained from (5.10). This implies that under a deformation with non-zero RR two form, the tensor $F_{\mu\nu}$ transforms by the projective transformation given in (5.10). The result in (5.10) is valid for a rank ≤ 6 RR two form. We also obtain from both (5.6) and (5.10) that the closed D1-brane metric only changes in the deformed 'directions', although the closed fundamental string metric changes in all 'directions'.

In section 2 the projective transformation of the tensor $E_{\mu\nu}$ was a consequence of combining $O(p+1,p+1)$ transformations, i.e., transformations in the T-duality group. Here instead the projective transformation of the tensor $F_{\mu\nu}$ can be seen as U-duality transformations. This is most easily seen by noticing that turning on, e.g., a rank 2 RR two form on the NS5-brane, can be seen from a fundamental string perspective to correspond to starting with an NS5-brane which is S-dualized to a D5-brane. Next, one deforms the D5-brane with a rank 2 B-field, using the $O(p+1,p+1)$ method, followed by S-duality. From a D1-brane perspective it is possible that the projective transformation of $F_{\mu\nu}$ can be seen to correspond to some kind of non-perturbative D1-brane 'T-duality' followed by a gauge transformation and finally a new non-perturbative T-duality in the directions of the gauge transformation of C_2. It would be interesting to investigate this non-perturbative 'T-duality' further, see section 6 for more comments. For related ideas concerning D1-brane 'T-duality', see [31].

We end this subsection by showing how the open D1-brane metric and noncommutativity parameter can be related to the closed D1-brane metric and the RR two form, similarly to how open and closed fundamental string data were related in section 2. We start by defining a tensor $\xi^{\mu\nu}$ as follows

$$\xi^{\mu\nu} = g^{-1} G^{\mu\nu}_{\mathrm{OD1}} - \frac{\Theta^{\mu\nu}_{\mathrm{OD1}}}{g\alpha'}\,. \tag{5.11}$$

Similarly to section 2, we obtain that F and ξ are related through

$$\xi^{\mu\nu} = (F^{-1})^{\mu\nu} . \tag{5.12}$$

This implies that the open D1-brane metric and noncommutativity parameter are manifestly deformation independent under deformations with an RR two form of an NS5-brane. It is interesting that we have found that several of the properties of how open and closed strings transform are the same for D1-branes and fundamental strings. Further, considering the conjectured $\mathrm{SL}(2,\mathbb{Z})$ symmetry of type IIB string theory we expect it to be possible to generalize (2.7) and (5.12) to (p,q) strings. We also expect it to be possible to generalize the $O(p+1,p+1)$ method (2.4) not only to deformations of NS5-branes with a RR two form (5.7) but to deform (p,q) 5-branes with a '(p,q)' theta deformation (i.e., turning on a combination of RR and NS-NS two forms). This can be obtained by performing an $\mathrm{SL}(2,\mathbb{Z})$ transformation of (2.4) with $p = 5$. This would be a different approach to deforming (p,q) 5-branes than in [25, 19], where instead the equations of motion were explicitly solved.

5.2 Deformation of NS5-branes with a RR three form

In this subsection we are going to generalize the results for deforming the type IIB NS5-brane to deformations of type IIA NS5-branes with a one parameter RR three form. We start with the following undeformed solution

$$\begin{aligned} ds^2 &= g^{(\mathrm{s})}_{\mu\nu} dx^\mu dx^\nu + g^{(\mathrm{s})}_{mn} dx^m dx^n , \qquad e^{2\phi} = g^2 F' , \\ g^2 B_6 &= -\omega dx^0 \wedge \cdots \wedge dx^5 \qquad B_2 = \gamma_2 . \end{aligned} \tag{5.13}$$

Next, generalizing the results for the type IIB NS5-brane is straight forward and gives

$$\begin{aligned} \tilde{g}^{(\mathrm{s})}_{\mu\nu} &= \left[\det\left(1 + \frac{1}{2}(\theta)^2\right)\right]^{1/6} g^{(\mathrm{s})}_{\mu\rho} \left[\left(1 + \frac{1}{2}(\theta)^2\right)^{-1}\right]^{\rho}{}_{\nu} , \quad \tilde{g}^{(\mathrm{s})}_{mn} = \left[\det\left(1 + \frac{1}{2}(\theta)^2\right)\right]^{1/6} g^{(\mathrm{s})}_{mn} , \\ \tilde{C}_3 &= \tilde{C}_{3\mathrm{a}} + \tilde{C}_{3\mathrm{b}} , \qquad \tilde{B}_2 = \gamma_2 , \qquad e^{2\tilde{\phi}} = e^{2\phi}\left[\det\left(1 + \frac{1}{2}(\theta)^2\right)\right]^{1/6} , \\ \tilde{B}_6 &= B_6 + \frac{1}{2}\tilde{C}_{3\mathrm{a}} \wedge \tilde{C}_{3\mathrm{b}} , \qquad \tilde{C}_5 = 0 , \end{aligned}$$

where

$$\begin{aligned} g\tilde{C}_{3\mathrm{a}} &= \frac{1}{6} g\tilde{C}^{3\mathrm{a}}_{\mu\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho , \quad g\tilde{C}^{3\mathrm{a}}_{\mu\nu\rho} = -g^2 e^{-2\phi} g^{(\mathrm{s})}_{\mu\rho'} g^{(\mathrm{s})}_{\sigma\nu} \theta^{\rho'\sigma\sigma'} g^{(\mathrm{s})}_{\sigma'\lambda} \left[\left(1 + \frac{1}{2}(\theta)^2\right)^{-1}\right]^{\lambda}{}_{\rho} , \\ g\tilde{C}_{3\mathrm{b}} &= \omega \frac{1}{6} \theta^{\mu\nu\rho} i_\mu i_\nu i_\rho dx^0 \wedge \cdots \wedge dx^5 . \end{aligned} \tag{5.14}$$

Here $\theta^{\mu\nu\rho}$ is a dimensionless (one parameter) anti-symmetric tensor, and $(\theta^2)^\mu{}_\nu$ is defined as in (4.6).

The above obtained formula (5.14) can be used to deform NS5-branes by turning on a non-zero RR three form. Below we show that if the above formula (5.14) first is T-dualized in a direction parallel to the deformed direction followed by S-duality, we obtain (2.4) with $p = 2$ and rank 2 B-field. This shows that (5.14) is correct. Further, for the half-supersymmetric case we have checked that electric, magnetic and light-like deformations,

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give the correct solutions corresponding to NS5-D2 and NS5-D2-D2-W bound states, respectively. These solutions in the particular form obtained here have been obtained before in [37, 26], using other methods. We note that similarly to the M5-brane case, electric and magnetic deformations give equivalent NS5-D2 solutions [37].

Also similarly to the eleven-dimensional case, the three form $\tilde{C}_3$ obeys a non-linear self-duality condition in the NS5-brane directions. Further, one can show that as in the M5-brane case, electric and light-like deformations give deformation independent open D2-brane metric and generalized noncommutativity parameter.[11]

Next, to show that (5.14) is correct, we are going to show that (5.14) is T-dual to (5.7). For T-duality we use the same conventions as was used in [26]. T-dualizing (5.14) in one of the deformed directions and identifying $\theta^{\mu\nu} = -\theta^{\mu\nu y}$, where y is the T-dualized direction, and using that

$$\frac{1}{2}(\theta^2)^\alpha{}_\beta \to -(\theta^2)^\alpha{}_\beta\,, \qquad \det\Big(1+\frac{1}{2}(\theta)^2\Big) \to [\det(1-(\theta)^2)]^{3/2}\,, \tag{5.15}$$

we obtain (5.14), as expected.

The solution generating technique (5.14) is related to (3.2) by a lift. We note that if (5.14) is lifted to eleven dimensions we obtain a slightly different version than (3.2) (there are minus signs which differ in a few places). However, the obtained formula is equivalent to (3.2). The reason for this difference lies in how (5.14) has been obtained. As is clear from above, (5.14) has been obtained by first S-dualizing the $O(p+1,p+1)$ method for $p=5$ and rank 2 B-field, followed by T-duality. We next give an example which illuminates and explains why there is a sign difference when lifting (5.14) to eleven dimensions, compared to (3.2).

Start with an M5-M2 solution which is smeared in the x^6 direction. The M5-brane is in the 1-5 directions and the M2-brane in the 1 and 5 directions. The M-theory three form is given by

$$A_3 = +A dx^0\wedge dx^1\wedge dx^5 - B dx^2\wedge dx^3\wedge dx^4 + \gamma_2\wedge dx^6\,, \tag{5.16}$$

while $\epsilon^{0123456789,10} = -1$. Note that in this example the exact form of the functions A and B is not interesting. Next, we compactify on a two torus in the 5 and 6 directions. This implies that we have the following parameters: ℓ_p, R_5 and R_6, where R_5 and R_6 are the radius in the 5 and 6 directions, respectively. We will now reduce in the 5 direction to type IIA and perform the following series of dualities: T-duality in the 6 direction, S-duality, T-duality in the 6 direction and finally we lift to M-theory (in the 5 direction). After performing these dualities we obtain an M5-M2 solution with the following three form:

$$A_3 = -A dx^0\wedge dx^1\wedge dx^6 - B dx^2\wedge dx^3\wedge dx^4 + \gamma_2\wedge dx^5\,, \tag{5.17}$$

while $\tilde{\ell_p} = \ell_p$, $\tilde{R}_5 = R_6$ and $\tilde{R}_6 = R_5$. This means that we have obtained an M5-M2 solution smeared in the 5 direction and where the M5-brane is in the 1-4 and 6 directions, while the M2-brane is in the 1 and 6 directions. What we have done here is a U-duality

[11] The open D2-brane metric and generalized noncommutativity parameter are essentially given by the same expressions as the open M2-brane data (3.8), see [28].

transformation which have switched place between the 5 and 6 directions. If we now let $5 \to 6$ and $6 \to 5$, (5.17) becomes

$$A_3 = -Adx^0 \wedge dx^1 \wedge dx^5 - Bdx^2 \wedge dx^3 \wedge dx^4 + \gamma_2 \wedge dx^6 \,. \qquad (5.18)$$

However, and most importantly, epsilon changes to $\epsilon^{0123456789,10} = +1$. The conclusion is that the sign change in one component of A_3 is compensated by the sign change in epsilon. This implies that if (5.16) solves the eleven-dimensional equations of motion, then so does (5.18), as long as also epsilon is changed. This explains why there is a sign difference when lifting (5.14) to eleven dimensions, compared to (3.2).

We will end this subsection by deriving an open D2-brane coupling G^2_{OD2} for an open D2-brane ending on an NS5-brane, using (5.14) and deformation independence. This open D2-brane coupling was first postulated in [13] (see also [37]) as the relevant coupling for the OD2-theory. Here we derive a covariant expression for this open D2-brane coupling which in the OD2 limit gives the coupling introduced in [13, 37]. We start with the following ansatz for the open D2-brane coupling

$$G^2_{\mathrm{OD2}} = e^{\tilde{\phi}} R(K) \,, \qquad (5.19)$$

where $R(K)$ is a function of

$$K = \sqrt{1 + \frac{1}{24}\tilde{C}^2} \,, \qquad \tilde{C}^2 = \tilde{g}^{\mu_1\nu_1}\tilde{g}^{\mu_2\nu_2}\tilde{g}^{\mu_3\nu_3}\tilde{C}_{\mu_1\mu_2\mu_3}\tilde{C}_{\nu_1\nu_2\nu_3} \,. \qquad (5.20)$$

Next, we are going to assume that the open D2-brane coupling is deformation independent for electric and light-like deformations (similar to the open D2-brane metric and generalized noncommutativity parameter). This implies that

$$G^2_{\mathrm{OD2}} = e^{\tilde{\phi}} R(K) = e^{\phi} \,, \qquad (5.21)$$

where e^{ϕ} is the undeformed dilaton. For a light-like deformation this is trivially satisfied for any $R(K)$. However, for an electric deformation (i.e., $\theta^{012} = \theta$, $\alpha, \beta = 0, 1, 2,$) we obtain, using (5.14) and (5.21), that

$$R(K) = h^{-1/4} \,, \qquad h = 1 + \theta^2 g^2 g^{\mathrm{D2}}_{(\alpha\beta)} \,, \qquad (5.22)$$

where $g^{\mathrm{D2}}_{(\alpha\beta)} = \det g^{\mathrm{D2}}_{\alpha\beta}$, and $g^{\mathrm{D2}}_{\alpha\beta} = e^{-\frac{2}{3}\phi} g^{(\mathrm{s})}_{\alpha\beta}$ is the undeformed closed D2-brane metric. Using that $0 < h < 1$ we get that $R(K) = [K(1 - \sqrt{1 - K^{-2}})]^{-1/2}$ (this expression can also be rewritten as $R(K) = [K(1 + \sqrt{1 - K^{-2}})]^{1/2}$). This leads to the following open D2-brane coupling

$$G^2_{\mathrm{OD2}} = e^{\tilde{\phi}} [K(1 - \sqrt{1 - K^{-2}})]^{-1/2} \,. \qquad (5.23)$$

As a check of (5.23) we have inserted the OD2-limit [13, 37] into (5.23) and found that it gives exactly the OD2 coupling defined in [13, 37]. It would be very interesting to derive this expression from a microscopic formulation of an open D2-brane ending on an NS5-brane. For a discussion of covariant expressions for open Dp-brane couplings ($p \neq 2$), see [38].

6. p-branes and $\mathrm{SL}(2,\mathbb{R})$

6.1 M-theory and $\mathrm{SL}(2,\mathbb{R})$

In this subsection we will argue that the metric and three form in the deformed 'directions', obtained from the solution generating technique (3.2), for a magnetic deformation, can be seen to emerge from certain $\mathrm{SL}(2,\mathbb{R})$ transformations of the three-torus Kähler structure parameter[12] (we choose e.g., $\theta^{345} = \theta$),

$$E = A_{345} + i\sqrt{\det g_{ab}}\,, \tag{6.1}$$

where $A_{345} = 0$, in the initial solution. As has been explained in section 3, when deforming an M5-brane one starts by compactifying the M5-brane on a three-torus etc., see section 3. This suggests that in order to obtain a new deformed solution we should perform the following $\mathrm{SL}(2,\mathbb{R})$ transformation of the Kähler structure parameter:

$$\tilde{E} = \frac{E}{-\theta E + 1} = \frac{-\theta \det g_{ab} + i\sqrt{\det g_{ab}}}{1 + \theta^2 \det g_{ab}}\,. \tag{6.2}$$

This implies that

$$\begin{aligned} \tilde{g}_{ab} &= \frac{g_{ab}}{(1 + \theta^2 \det g_{ab})^{2/3}}\,, \\ \tilde{A}_{345} &= \frac{-\theta \det g_{ab}}{1 + \theta^2 \det g_{ab}}\,. \end{aligned} \tag{6.3}$$

Note that we can only expect the transformation (6.2) to be valid for a magnetic deformation, since electric and light-like deformations involve the time direction. However, the deformed metric and three form can be obtained from (6.3) also for electric deformations due to analytic continuation from the magnetic case. Comparing (6.3) with the metric and three form (in the deformed 'directions') in (3.2) we obtain that they are identical for magnetic (and electric) deformations. For the magnetic deformations the above means that the metric and three form in the deformed 'directions' transform 'together' under the above $\mathrm{SL}(2,\mathbb{R})$ transformation, when deforming the M5-brane.

The above $\mathrm{SL}(2,\mathbb{R})$ transformation is easily seen to be given by

$$\tilde{E} = (\mathrm{S}^{-1}\mathrm{T}\mathrm{S})E\,, \tag{6.4}$$

where

$$\mathrm{T} = \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix}, \qquad \mathrm{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{6.5}$$

Note that a general $\mathrm{SL}(2,\mathbb{R})$ transformation, i.e., $\tau \to \frac{a\tau+b}{c\tau+d}$, is given by the following matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{6.6}$$

where $ad - bc = 1$.

[12] For related issues see [31, 32].

The transformation given in (6.4) implies that we first invert the volume of the three-torus (i.e., we use the S transformation on the Kähler structure parameter[13]) followed by a gauge transformation and finally, we invert the volume of the torus again. Since the $\mathrm{SL}(2,\mathbb{R})$ transformations we performed here gave the same result as in (3.2), for the magnetic case, we have yet another non trivial test of (3.2). Also, viewing the deformations as certain $\mathrm{SL}(2,\mathbb{R})$ transformations makes the deformation procedure more transparent as U-duality transformations. We note that $\mathrm{SL}(2,\mathbb{R})$ is part of the U-duality group $\mathrm{SL}(3,\mathbb{R})\times \mathrm{SL}(2,\mathbb{R})$ for eleven-dimensional supergravity compactified on a three-torus [39]. In the full M-theory the U-duality group is expected to be $\mathrm{SL}(3,\mathbb{Z})\times\mathrm{SL}(2,\mathbb{Z})$ [39].

It is easy to see that (6.4) is the only non-trivial deformation which is possible to perform if there has to be an equal number of S and S^{-1}. The reason for the equal number of S and S^{-1} is because S can be viewed as 'inverting' the three torus, while S^{-1} is just the inverse of this transformation. Therefore, an equal number of S and S^{-1} implies that starting with an M5-brane we end with a (deformed) M5-brane. Note that an odd total number of S and S^{-1} transformations would imply that we end with an M2-brane solution.

The next step towards a further understanding of (3.2), would be to obtain exactly how the tensors ($g_{\mu\nu}$ and $A_{\mu\nu\rho}$) transform and not only the complex scalar (E), since this should lead to a better understanding about also the electric and light-like cases. This is in contrast to the string case (see section 2 and 5) where we know how the *tensors* $E_{\mu\nu}$ and $F_{\mu\nu}$ transform. For example, is it (in the M-theory case) possible to form a three index tensor out of the metric and three form which transforms in some 'nice' way? This three index tensor should be possible to obtain since we know from (3.2) the end result of this transformation (i.e., from (3.2) we obtain the end result for $g_{\mu\nu}$ and $A_{\mu\nu\rho}$). This new three index tensor should also be the generalization of the two index tensor $E_{\mu\nu}=g_{\mu\nu}+B_{\mu\nu}$ to eleven dimensions. So far, we have unfortunately not been able to obtain this three index tensor. We plan to discuss this further in a future paper [33].

6.2 D-branes and $\mathrm{SL}(2,\mathbb{R})$

In this subsection we show how the metric and $(p+1)$-form C ($p=0,1,2,3,4$) for a D$(p+2)$-brane or NS5-brane, transform in the deformation directions, under a magnetic deformation with a constant anti-symmetric $(p+1)$-form θ^{p+1}. Similarly to the M-theory case in the last subsection we start by giving the Kähler structure parameter for a NS5-brane or a D$(p+2)$-brane compactified on a $(p+1)$-torus $(a,b=1,2,\ldots,p+1)$

$$E = gC_{12\ldots(p+1)} + ig\sqrt{\det g^{\mathrm{D}p}_{ab}}\,, \tag{6.7}$$

where g is the closed string coupling constant and $C_{12\ldots(p+1)}=0$, in the initial solution. To obtain a new deformed solution we perform the same $\mathrm{SL}(2,\mathbb{R})$ transformation (6.4) as we did in the M-theory case. This gives the following result

$$\tilde{g}^{\mathrm{D}p}_{ab} = \frac{g^{\mathrm{D}p}_{ab}}{(1+\theta^2 g^2 \det g^{\mathrm{D}p}_{ab})^{\frac{2}{p+1}}}\,,$$

[13] In [31] this transformation was named 'T-duality' for M2-branes, because it exchanges the Kaluza-Klein modes with the wrapping modes of the M2-brane.

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$$g\tilde{C}_{12\ldots(p+1)} = \frac{-\theta g^2 \det g_{ab}}{1+\theta^2 g^2 \det g_{ab}}\,. \tag{6.8}$$

For $p = 2$ we compare (6.8) with the metric and three form (in the theta 'directions') in (5.14) and (4.5), for a magnetic deformation, and obtain that they are identical. For $p = 1$ we also obtain perfect agreement if we let $\theta \to -\theta$ in (6.8).

We note that, as expected, for both electric and magnetic deformations, the open Dp-brane metric is deformation independent and the generalized noncommutativity parameter is shifted by a constant $\sim \theta$. It is interesting to note that since the open Dp-brane metric is deformation independent and the generalized noncommutativity parameter is shifted by a constant $\sim \theta$, when E transforms according to (6.4), the transformation of the open Dp-brane data (in the deformed direction) can be written as follows

$$\tau \to \mathrm{T}\tau\,. \tag{6.9}$$

Here

$$\tau = ig^{-1}\sqrt{\det(G)^{-1}_{\widetilde{\mathrm{OD}p}}} + \frac{\Theta_{\widetilde{\mathrm{OD}p}}}{g\alpha'^{\frac{p+1}{2}}}\,, \tag{6.10}$$

while $\det(G)^{-1}_{\widetilde{\mathrm{OD}p}}$ is the determinant of the inverse of the open Dp-brane metric and $\Theta_{\widetilde{\mathrm{OD}p}}$ is the open Dp-brane generalized noncommutativity parameter. Note that before the deformation $\tau = (\mathrm{S})E$, i.e., (6.9) implies that $\tau \to \tau = (\mathrm{TS})E$, since $\Theta_{\widetilde{\mathrm{OD}p}} = \theta g\alpha'^{\frac{p+1}{2}}$. This means that when the closed Dp-brane data (i.e., $C_{\mathrm{p}+1}$ and $g^{\mathrm{D}p}_{ab}$ in the deformed directions) transform as in (6.4), the open Dp-brane data (metric and theta parameter in the deformed directions) have the simple transformation (6.9). Further, after the deformation the new τ can be seen to be

$$\tilde{\tau} = \mathrm{S}\tilde{E}\,. \tag{6.11}$$

Comparing (6.11) with the open Dp-brane data obtained in [28], we find that (6.11) gives the correct open Dp-brane metric and generalized noncommutativity parameter (in the deformed directions).

It is interesting that the relation between the open Dp-brane data and 'closed' Dp-brane data in (6.11) is very similar to how the open and closed string (D1-brane) data are related in (2.7) (and (5.12)). We do not yet know how relevant this result is. However, it is probably yet another indication that the open Dp-brane metrics and generalized noncommutativity parameters, obtained in [28], are correct.

7. Conclusions

In this paper we have obtained an M-theory solution generating technique, which can be used to deform an M5-brane with a non-zero three form A. To check that this solution generating technique gives the correct results we have performed several tests, both directly in eleven dimensions (see section 3) and by reduction to ten dimensions and comparing with known results (see section 4). For example, we showed that the eleven-dimensional 'tensor' equation is satisfied and that (3.2) gives the correct half-supersymmetric bound state solutions M5-M2 and M5-M2-M2-MW. Together, the tests that we have performed in ten and eleven dimensions, strongly indicate that we have obtained the correct solution generating technique. Note, however, that we have not rigorously proved that (3.2) is correct. A rigorous proof would be to also show that the eleven-dimensional 'Einsteins' equation of motion is satisfied by (3.2). This seems difficult to show, because 'Einsteins' equation of motion is second order in derivatives of the deformed metric. This is an important difference compared to the eleven-dimensional 'tensor' equation, which avoids explicit derivatives on the deformed metric.

In section 6 we have shown that for a magnetic deformation, it is possible to view the deformation as certain $SL(2,\mathbb{R})$ transformations of the Kähler structure parameter for the three-torus, on which the M5-brane has been compactified. This is an important result since it shows that (as expected) deforming an M5-brane corresponds to performing the appropriate U-duality transformations,[14] i.e., it is correct to view the deformation procedure as first performing (an M-theory version of) 'T-duality' in three directions, followed by a gauge transformation, and finishing with 'T-duality' in the same three directions as before.

We have also in this paper obtained solution generating techniques for deforming the type IIA/B NS5-branes with a one parameter RR three or two form, respectively. These solution generating techniques were shown to generate the expected results. Further, in the type IIA case we used the newly obtained solution generating technique and deformation independence to derive a covariant expression for an open D2-brane coupling, relevant for OD2-theory. It would be very interesting if this result could be derived from a microscopic formulation of an open D2-brane ending on an NS5-brane.

In a future paper [33] we plan to expand on the results in section 6 to tensor relations for $A_{\mu\nu\rho}$ and $g_{\mu\nu}$, in order to obtain a better understanding of M-theory 'T-duality'. For example, it would be interesting to see if it is possible to derive some kind of 'generalization' of the relation $E_{\mu\nu} = g_{\mu\nu} + B_{\mu\nu}$ in string theory to M-theory? Our two main motivations for a further study of M-theory 'T-duality', is to obtain more information about the open membrane metric and generalized noncommutativity parameter, which are deformation independent under certain M-theory 'T-duality' transformations (i.e., under the combination 'T-duality' + gauge transformation + 'T-duality', see [28] and section 3 and 6 for more details), and to also be able to derive (i.e., to rigorously prove) the solution generating technique given in (3.2), from a microscopic formulation of M-theory 'T-duality'.

Acknowledgments

We are grateful to R. Argurio, P. Arvidsson, M. Cederwall, U. Gran, M. Nielsen, B.E.W. Nilsson and P. Sundell for valuable discussions and comments.

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